

## Shear rate equations for yield stress fluids in couette flow

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**Abstract.** The accuracy of the Krieger-Maron, power law and Yang-Krieger equations for calculating the shear rate of three specific Herschel-Bulkley fluids ( $\tau = \tau_y + K\dot{\gamma}^n$ ) in couette flow within the annulus of a concentric cylinder was assessed. The chosen fluids have a shear rate exponent  $n$  of 1, 1/2 and 1/3. The accuracy for all three equations was found to depend upon the fluid model, stress level and radius or gap ratio  $\epsilon$  of the concentric cylinders. With the Krieger-Maron equation, the shear rate is only accurate in the fully sheared regime for all  $n$  and  $\epsilon$ . In contrast both the power law and Yang-Krieger equations are accurate in both the plug and fully sheared flow regimes. For both equations, the largest absolute error generally decreases with increasing  $\epsilon$  and decreasing  $n$  and is usually located near the transition from plug to fully sheared flow. In terms of the largest absolute error, the Yang-Krieger is more accurate.

**Abstrak.** Kejituan persamaan-persamaan Krieger-Maron, hukum kuasa dan Yang-Krieger untuk mengirakan kadar ricih bagi tiga bendalir spesifik Herschel-Bulkley ( $\tau = \tau_y + K\dot{\gamma}^n$ ) dalam aliran 'couette' di dalam anulus silinder konsentrik telah dinilai. Bendalir-bendalir yang terpilih mempunyai eksponen model bendalir,  $n$ , 1, 1/2 dan 1/3. Kejituan ketiga persamaan itu bergantung kepada model bendalir, aras tegasan dan nilai nisbah ruang atau jejari silinder konsentrik,  $\epsilon$ . Bagi persamaan Krieger-Maron, terdapat kiraan kadar ricih yang hanya tepat dalam regim aliran ricih sepenuhnya untuk semua nilai  $n$  dan  $\epsilon$ . Secara bandingan, terdapat persamaan-persamaan hukum kuasa dan Yang-Krieger yang tepat dalam kedua regim aliran, penyumbat dan ricih sepenuhnya. Bagi kedua persamaan ini terdapat nilai maksima ralat mutlak secara amnya berkurang apabila nilai  $\epsilon$  bertambah dan apabila  $n$  berkurang, dan nilai maksima ini biasanya berlaku didalam peralihan antara aliran penyumbat dan ricih sepenuhnya. Dari segi nilai maksima ralat mutlak, persamaan Yang-Krieger adalah lebih tetap berbanding dengan persamaan hukum biasa.

### Introduction

Industrial suspensions such as coal slurries, food hydrocolloids, paints and toothpastes typically display non-Newtonian flow characteristics and often with a yield stress. Their flow characteristics are best described by the three-parameter Herschel-Bulkley model

( $\tau = \tau_y + K\dot{\gamma}^n$ ) which incorporates an additional variable shear rate exponent,  $n$ , that is not found in the classical Bingham model. The Herschel-Bulkley (HB) model has been used successfully to describe the flow behaviour of flocculated dispersions [1-3]. The flow behaviour is often characterised using a concentric cylinder viscometer. The true shear rate however can only

be estimated using one of a number of equations. Most of these equations were attributed to Krieger and co-workers [4-8]. Before these equations can be used their accuracy must be assessed with model fluids. Here, the accuracy of three shear-rate equations was assessed with three specific HB models in couette cylinder flow.

The Herschel-Bulkley equation is given by:

$$\dot{\gamma} = f(\tau) = \left( \frac{\tau - \tau_y}{K} \right)^{\frac{1}{n}} \quad (1)$$

where  $\tau_y$ ,  $K$  and  $n$  the yield stress, consistency index and shear rate exponent. In this study  $n$  of 1, 1/2 and 1/3 were chosen. This range of  $n$  would normally cover the entire range of flow behaviour exhibited by most yield stress fluids [3]. When  $n$  is 1, it is a Bingham fluid.

The three shear rate equations assessed were:

1) the Power law [6]:

$$\dot{\gamma} = \frac{2\Omega S^{-1}}{(1 - \varepsilon^{-2/S})} \quad (2)$$

where

$\Omega$  the rotational speed,

$\varepsilon$  the radius ratio of the outer to the inner cylinder,

2) the Krieger-Marom [5]:

$$\dot{\gamma} = \frac{2\Omega}{(1 - \varepsilon^{-2})} \left( b_0(\varepsilon) + \left(\frac{1}{S} - 1\right) b_1(\varepsilon) + \left( \left(\frac{1}{S} - 1\right)^2 - \frac{S'}{S^3} \right) b_2(\varepsilon) \right) \quad (3)$$

where

$$b_0(\varepsilon) = \frac{(\varepsilon^2 - 1)}{\varepsilon^2} \left( \frac{1}{2} + \frac{1}{2 \ln \varepsilon} + \frac{\ln \varepsilon}{6} \right),$$

$$b_1(\varepsilon) = \frac{(\varepsilon^2 - 1)}{2 \varepsilon^2} \left( 1 + \frac{2}{3} \ln \varepsilon \right),$$

$$b_2(\varepsilon) = \frac{(\varepsilon^2 - 1)}{6 \varepsilon^2} \ln \varepsilon,$$

and

3) the Yang-Krieger [8]:

$$\dot{\gamma} = \frac{2\Omega S^{-1}}{(1 - \varepsilon^{-2/S})} \left( 1 - \frac{S'}{S} C_1(\delta) + \left( 3 \left( \frac{S'}{S} \right)^2 - \frac{S''}{S} \right) C_2(\delta) + 3 \left( \frac{S'}{S} \right)^3 C_3(\delta) \right) \quad (4)$$

where

$$\delta = \frac{2}{S} \ln \varepsilon,$$

$$C_1(\delta) = \frac{\delta}{2} (e^\delta - 1)^{-2} (\delta e^\delta - 2e^\delta + \delta + 2),$$

$$C_2(\delta) = \frac{\delta^2}{6} (e^\delta - 1)^{-3} (3e^{2\delta} - \delta e^{2\delta} - 4\delta e^\delta - \delta - 3),$$

$$C_3(\delta) = \frac{\delta^3}{24} (e^\delta - 1)^{-4} (\delta e^{3\delta} - 4e^{3\delta} + 11\delta e^{2\delta} - 12e^{2\delta} + 12e^\delta + 11\delta e^\delta + \delta + 4)$$

and  $S = d \ln \tau / d \ln \Omega$ ,  $S' = dS / d \ln \Omega$

and  $S'' = dS' / d \ln \Omega$ .  $\dot{\gamma}$  becomes  $\gamma$  for shear rate at the inner bob where  $\tau_1$  is measured. The Yang-Krieger [8] equation is a truncated form of the Euler-MacLaurin solution expressed as a series of summable subseries [6,7]. The leading term is the power law expression given by equation (2).

#### Theory and methodology

For a fluid sheared in the annulus of a concentric cylinder, the shear rate  $\dot{\gamma}$  is implicitly given by:

$$\Omega = - \frac{1}{2} \int_{\tau_1}^{\tau_2} \dot{\gamma} \frac{d\tau}{\tau} \quad (5)$$

where  $\tau_2$  is the shear stress at the outer cylinder.

Equation (5) can only be solved exactly for  $\dot{\gamma}$ , from the measured  $\Omega$  and  $\tau$  data, if the fluid model is known apriori. Hence, this equation is of little use as the rheological behaviour of most materials is usually unknown. Note that numerical techniques with curve fitting, smoothing and other

capabilities had been used to solve equation (5) for relatively simple fluids [9-11].

There are two flow regimes, plug and fully sheared flow, for a yield stress fluid sheared in the annular of a concentric cylinder. For an inner rotating cylinder, plug flow occurs when

$$\tau_2 < \tau_y < \tau_1,$$

only that part of the fluid adjacent to the rotating cylinder experiences shear flow. The plug flow boundary conditions for equation (5) are

$$\tau_1 = \tau_1 \text{ and } \tau_2 = \tau_y.$$

The computation of  $\gamma_1$  using equations (2)-

(4) requires the determination of  $S, S'$  and  $S''$  at each  $\tau_1$  from the logarithmic plot of  $\tau_1$  versus  $\Omega$  or alternatively from an analytical  $\Omega - \tau_1$  relationship which can be obtained by integrating equation (5) with a HB fluid model. All relationships between  $\Omega$  and  $\tau_1$  for both plug and fully sheared flow for the three HB fluids are tabulated in Table 1. Further manipulation is required to obtain the values for  $S, S'$  and  $S''$  from  $\Omega - \tau_1$  equations and the mathematics involved are listed in Appendix A.

**Results and discussion**

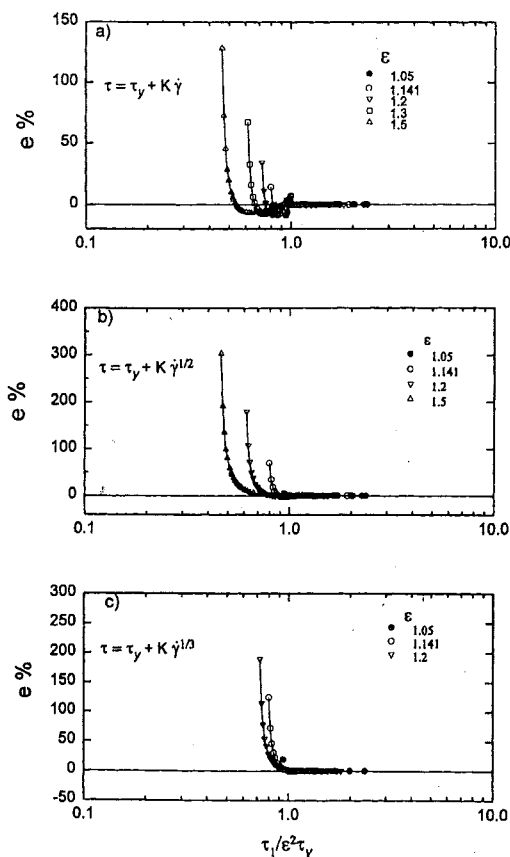
The accuracy of the Krieger-Marom, power law and Yang-Krieger equations (respectively given by equations (2), (3) and (4)) was determined by comparing the calculated shear rate with the exact value obtained for the HB model. Their difference is expressed in terms of a percentage error  $e\%$  which is given by:

$$e\% = \frac{\gamma_1 - \gamma_o}{\gamma_o} \times 100$$

where  $\gamma_o$  and  $\gamma_1$  the model and calculated shear rate. The percentage error was evaluated as a function of a dimensionless shear stress,  $\tau_1/(\epsilon^2\tau_y)$ . One advantage for using  $\tau_1/(\epsilon^2\tau_y)$  is a well defined

flow regime. For instance when  $\tau_1/(\epsilon^2\tau_y) < 1.0$  it is plug flow. A further advantage is the  $e\%$  versus  $\tau_1/(\epsilon^2\tau_y)$  plot being insensitive to the magnitude of  $\tau_y$  and  $K$ .

The shear rate error  $e\%$  obtained using the Krieger-Marom equation plotted as a function of dimensionless shear stress,  $\tau_1/(\epsilon^2\tau_y)$ , for HB fluids with exponent  $n$  of 1, 1/2 and 1/3 is shown in Figure 1. The radius or gap ratio  $\epsilon$  evaluated ranges from 1.05 to 1.5. In the plug flow regime, the error increases exponentially with decreasing  $\tau_1/(\epsilon^2\tau_y)$  for all fluids and  $\epsilon$ . In the fully sheared regime, the error rarely exceeds 1% for all fluids and  $\epsilon$ .



**Figure 1.** The percentage shear rate error versus dimensionless shear stress obtained using the Krieger-Marom equation for Herschel Buckley fluids with shear rate exponent of a)  $n = 1$ , b)  $n = 1/2$  and c)  $n = 1/3$  and for a range of gap ratio  $\epsilon$ .

The Krieger-Marom equation is useful provided that plug flow can be avoided. This can

be achieved by using a small gap viscometer as plug flow only occurs within a narrow  $\tau_1/(\varepsilon^2\tau_y)$  range at small  $\varepsilon$ . However such a viscometer is not suitable for suspensions especially when the particle size is comparable with the gap width.

A similar plot of shear rate error versus  $\tau_1/(\varepsilon^2\tau_y)$  obtained for the power law equation is shown in Figure 2. The power law exponent  $S$  was determined at each point on the  $\tau$  versus  $\Omega$  curve.

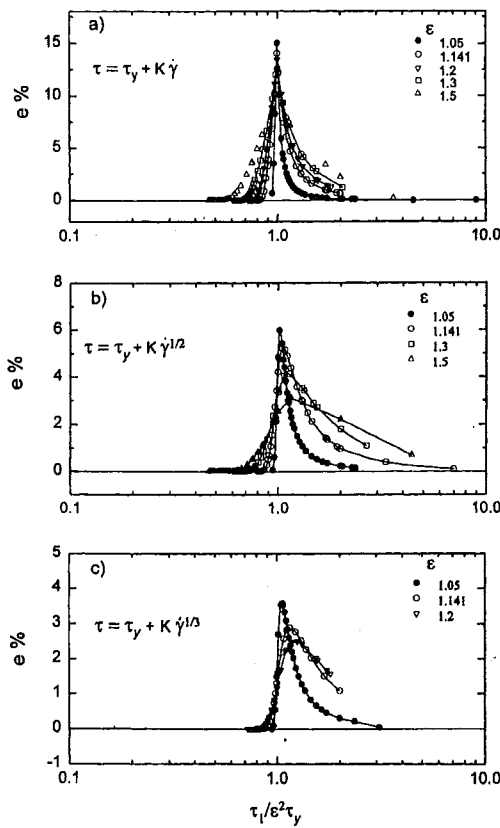


Figure 2. The shear rate error versus dimensionless shear stress obtained using the power law expression for Herschel-Buckley fluids with shear rate exponent of a)  $n = 1$ , b)  $n = 1/2$  and c)  $n = 1/3$  and for a range of  $\varepsilon$ .

A number of interesting features common to all three fluids were observed. The error is always positive. There is distinct maximum error located in the neighbourhood of the transition from plug to fully sheared flow. This maximum generally decreases with increasing radius ratio. For example, when  $\varepsilon$  was increased from 1.05 to 1.5,

the maximum decreased from 16% to 12% for  $n$  of 1, 6% to 3% for  $n$  of 1/2 and 3.5% to 0% for  $n$  of 1/3. The maximum also decreases with decreasing  $n$ . For example, it is 16% for  $n$  of 1.0, 6% for  $n$  of 1/2 and 3.5% for  $n$  of 1/3 at  $\varepsilon$  of 1.05. It is clear that the error in the plug flow regime is much smaller than that obtained using the Krieger-Maron equation.

Darby [12] has assessed the accuracy of the power law equation for the Bingham and Casson ( $\tau^{1/2} = \tau_y^{1/2} + C\dot{\gamma}^{1/2}$ ) fluids and reported a maximum error of 14% and 6% for  $\varepsilon$  of 1.1. A similar maximum of 14% for Bingham fluid for  $\varepsilon$  of 1.141 was obtained here. For  $n$  of 1/2, the HB fluid with the same exponent as the Casson, a similar maximum error of 6% was also obtained for  $\varepsilon$  of 1.141. This result suggests that the magnitude of the error is solely determined by the shear rate exponent.

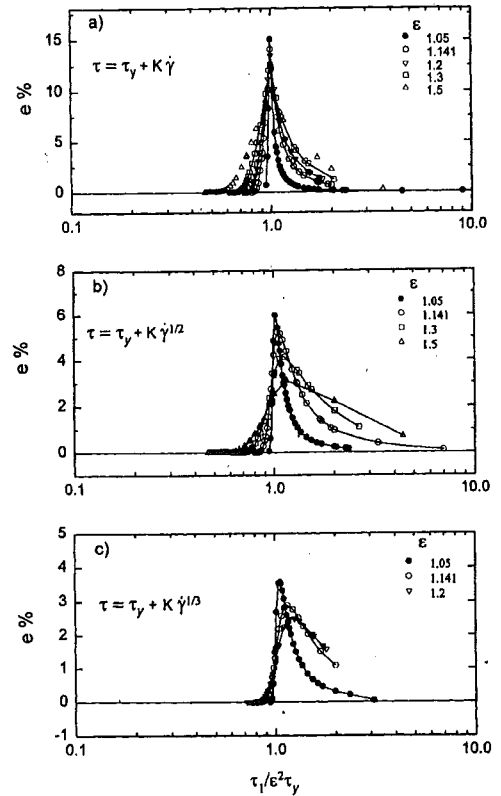


Figure 3. The shear rate error versus dimensionless shear stress obtained using the Yang-Krieger expression for Herschel-Buckley fluids with shear rate exponent of a)  $n = 1$ , b)  $n = 1/2$  and c)  $n = 1/3$  and for a range of  $\varepsilon$ .

The results for the Yang-Krieger equation are shown in Figures 3. The results show an oscillating  $e\%$  about the 0% baseline. The largest absolute  $e\%$  is located near the flow transition. This largest  $e\%$  generally increases with increasing  $n$  for a given  $\epsilon$ . It is also small. For example it is less than 2% for  $n$  of 1/2 and 1/3 at  $\epsilon$  of between 1.05 and 1.5. However, it is 8% for  $n$  of 1.0 at  $\epsilon$  of 1.141 and decreases 5% at  $\epsilon$  of 1.5. The error is also relatively small in the plug flow regime.

Figure 4 shows the plot of  $e\%$  versus  $\tau_1/(\epsilon^2\tau_y)$  for all three fluids at a fixed  $\epsilon$  of 1.141 where the Newtonian expression is used to specify the shear rate. The power law becomes the Newtonian expression when  $S$  is 1.0. The errors are clearly large. Hence, caution is required when using any commercial concentric cylinder viscometer which usually quote shear rate factors based on the Newtonian or narrow gap assumption. Note that the gap ratio of 1.05 and 1.141 are used in our laboratory.

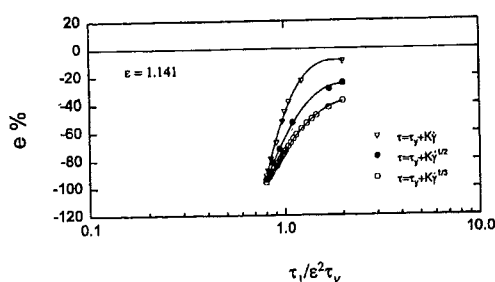


Figure 4. The shear rate error versus dimensionless shear stress obtained using the Newtonian expression for the three Herschel Buckley fluids and for  $\epsilon = 1.141$ .

The well-known Krieger-Maron equation is flow regime dependent. It cannot be used in the plug flow regime. For it to be useful, the yield stress needs to be determined independently so as to establish the flow regime [13]. In contrast both the power law and the Yang-Krieger equations can be used in both flow regimes. As a note of caution, the Yang-Krieger equation can be subjected to large error for real fluids. This is because the higher order differentiation of the experimental data to get  $S'$  and  $S''$  can be highly inaccurate. In contrast the power law requires only the first

order derivative  $S'$ . If a maximum error of 10 to 15% is acceptable then the power law equation should be used for real fluids. Furthermore the maximum error can be further reduced to less than 10% by using a large gap,  $\epsilon > 1.5$ .

The most accurate equation for calculating the shear rate of Herschel-Bulkley fluids in terms of the absolute value of the largest error and the absence of flow regime dependence is the Yang-Krieger. However this equation requires higher order differentiation of the experimental data. Although the power law equation is less accurate but it requires only the first order derivative. The use of a large gap viscometer can reduce the error considerably for both equations. The Krieger-Maron equation is extremely accurate in the fully sheared regime but suffers from large inaccuracy in the plug flow regime.

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**Table 1.**  $\Omega$ - $\tau_1$  relationships for plug and fully sheared flow obtained for the three Herschel-Bulkley fluids.

$$\tau = \tau_y + K \gamma$$

Plug flow:  $\Omega = \frac{1}{2K} \left( \tau_1 - \tau_y + \tau_y \ln \frac{\tau_y}{\tau_1} \right)$

Fully sheared flow:  $\Omega = \frac{1}{2K} \left( \tau_1 (1 - \varepsilon^{-2}) + \tau_y \ln \varepsilon^{-2} \right)$

$$\tau = \tau_y + K \gamma^{1/2}$$

Plug flow:  $\Omega = \frac{1}{2K^2} \left( \frac{\tau_1^2}{2} - 2\tau_y \tau_1 + \tau_y^2 \ln \frac{\tau_1}{\tau_y} + \frac{3}{2} \tau_y^2 \right)$

Fully sheared flow:  $\Omega = \frac{1}{2K^2} \left( \frac{\tau_1^2}{2} (1 - \varepsilon^{-4}) - 2\tau_y \tau_1 (1 - \varepsilon^{-2}) + \tau_y^2 \ln \varepsilon^2 \right)$

$$\tau = \tau_y + K \gamma^{1/3}$$

Plug flow:  $\Omega = \frac{1}{2K^3} \left( \frac{\tau_1^3}{3} - \frac{3}{2} \tau_y \tau_1^2 + 3\tau_y^2 \tau_1 - \tau_y^3 \left( \frac{11}{6} - \ln \frac{\tau_y}{\tau_1} \right) \right)$

Fully sheared flow:  $\Omega = \frac{1}{2K^3} \left( \frac{\tau_1^3}{3} (1 - \varepsilon^{-6}) - \frac{3}{2} \tau_y \tau_1^2 (1 - \varepsilon^{-4}) + 3\tau_y^2 \tau_1 (1 - \varepsilon^{-2}) - \tau_y^3 \ln \varepsilon^2 \right)$

Appendix A. Mathematical relationship for obtaining S, S' and S'' from the  $\Omega$ - $\tau_1$  relationship.

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$$S = \frac{d \ln \tau_1}{d \ln \Omega} = \frac{\Omega}{\tau_1} \frac{d \tau_1}{d \Omega}$$

$$S' = \frac{d^2 \ln \tau_1}{d \ln \Omega^2} = \frac{\Omega^2}{\tau_1} \left( \frac{d^2 \tau_1}{d \Omega^2} + \frac{1}{\Omega} \frac{d \tau_1}{d \Omega} - \frac{1}{\tau_1} \left( \frac{d \tau_1}{d \Omega} \right)^2 \right)$$

where

$$\frac{d^2 \tau_1}{d \Omega^2} = - \frac{1}{(d \Omega / d \tau_1)^3} \frac{d^2 \Omega}{d \tau_1^2}$$

$$S'' = \frac{d^3 \ln \tau_1}{d \ln \Omega^3} = \Omega \left( \frac{1}{\tau_1} \frac{d \tau_1}{d \Omega} - \frac{3 \Omega}{\tau_1^2} \left( \frac{d \tau_1}{d \Omega} \right)^2 + \frac{2 \Omega^2}{\tau_1^3} \left( \frac{d \tau_1}{d \Omega} \right)^3 + \frac{3 \Omega}{\tau_1} \left( \frac{d^2 \tau_1}{d \Omega^2} \right) - \frac{3 \Omega^2}{\tau_1^2} \left( \frac{d \tau_1}{d \Omega} \frac{d^2 \tau_1}{d \Omega^2} \right) + \frac{\Omega^2}{\tau_1} \left( \frac{d^3 \tau_1}{d \Omega^3} \right) \right)$$

where

$$\frac{d^3 \tau_1}{d \Omega^3} = - \frac{d \tau_1}{d \Omega} \left( \frac{-3}{(d \Omega / d \tau_1)^4} \left( \frac{d^2 \Omega}{d \tau_1^2} \right)^2 + \frac{1}{(d \Omega / d \tau_1)^3} \frac{d^3 \Omega}{d \tau_1^3} \right)$$

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