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Direct Numerical Algorithm as First, Second and Third-Order IVPS Solver

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Abstract: This paper presents a novel numerical model for accurately integrating initial value problems of multi-order ordinary differential equations (ODEs) of first, second, and third orders. Chebyshev polynomials are employed as the basis functions, and the collocation technique is used to develop continuous schemes that are evaluated at selected points to formulate the proposed multi-order ODE solver, which is applied in a block-by-block manner. The convergence analysis is carried out to establish the zero-stability and consistency of the method. Comparisons with existing methods show the superior performance of the proposed method. The results indicate its ability to solve multi-order ODEs more effectively while reducing the computational cost. This work represents a significant advance in numerical integration for ODEs, providing improved accuracy and efficiency in solving a wide range of multi-order ODE problems.

Keywords: *Block method, collocation, consistency, convergence, zero-stability.*

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1. Introduction

This paper aims to develop a linear multistep method of the form

$$\left\{ y(x) = h\left(\sum_{r=0}^{s}\beta_r(t)m_{n+r}\right) + h^2\left(\sum_{r=0}^{s}\mu_r(t)l_{n+r}\right) + h^3\left(\sum_{r=0}^{s}\delta_r(t)k_{n+r}\right) \right. \tag{1}$$

for the direct integration of ordinary differential equations (ODEs), where in (1) either of $\sigma_0(t)$ and $\beta_0(t)$ do not vanish, $\sigma_s(t)=1$, $\beta_s(t)\neq 0$ and $s=1$.

Our focus is on three distinct types of ordinary differential equations (ODEs), denoted Equation (2) in this paper.

$$\omega'(x) = f(x,\omega(x)), \omega(x_0) = \omega_0 \tag{2a}$$

$$\omega''(x) = f(x,\omega(x),\omega'(x)), \omega(x_0) = \omega_0, \omega'(x_0) = \omega_0' \tag{2b}$$

$$\omega'''(x) = f(x,\omega(x),\omega'(x),\omega''(x)), \omega(x_0) = \omega_0, \omega'(x_0) = \omega_0', \omega''(x_0) = \omega_0'' \tag{2c}$$

Numerical methods have become essential tools for approximating solutions to differential equations, particularly for real-world problems whose analytical solutions do not exist (Adeyefa & Kuboye, 2020; Jator, 2010). As a result, there is a growing need for more efficient and accurate computational methods to solve such initial value ordinary differential equation (ODE) problems.

Many authors have proposed different numerical methods for solving Equation (2a), as documented in Ajileye et al. (2018). Conventionally, the solutions of Equations (2b) and (2c) are obtained by reducing and into equivalent systems of first-order ODEs. However, this approach often increases computational time and effort (Henrichi, 1962; Lambert, 1973). In contrast, direct integration methods for Equation (2b) have

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been proposed (Awoyemi, 1999; Fatunla, 1991; Koboye, 2015), employing self-starting numerical algorithms that remove the need for predictors. Equation (2c) has received considerable attention from researchers, leading to the development of various schemes (Allogmany, 2020; Hussain, 2017; Olabode, 2009; Rufai, 2023, Duromola, 2022, Rufai et al., 2023, Rufai et al., 2024) for solving third-order ODEs. In addition, Ramos & Rufai (2021) introduced a one-step Lobatto-type hybrid block method incorporating a second derivative to obtain approximate solutions in an embedded-like form executed in adaptive mode, thereby improving the performance of higher-order block methods.

Notably, a recent trend among scholars is the development of a unified scheme for solving equations of the form given in Equation (2). This emerging area of research has attracted the interest of many investigators. Matthew et al. (2024) formulated a single numerical method for the integration of first- and second-order ordinary differential equations. Adeyefa et al. (2020) proposed a method for the direct integration of second- and third-order ordinary differential equations. Abolarin et al. developed implicit hybrid block methods for solving second-, third, and fourth-order ordinary differential equations directly using power series. Given the elegant properties of Chebyshev polynomials, the proposed method employs them as basis functions.

The main novelty of this work is the formulation of a two-step hybrid block method (THBM), a single numerical model that can efficiently and accurately handle first-, second-, and third-order ODEs. Section 2 sets out the formulation of the continuous algorithm that gives rise to the discrete method for solving Equation (2). The convergence analysis of the proposed method is presented in Section 3, followed by the results in Section 4. Finally, Section 5 provides concluding remarks.

2. Development of Two-Step Hybrid Block Method (THBM)

In this section, THBM is derived using the well-known Chebyshev polynomial of the first degree, given by

$$y(x)=\sum_{r=0}^{s+9}\sigma_r X_r \tag{3}$$

where X^r are the parameters of Chebyshev polynomials and σ_r are unknown constants to be determined. The Chebyshev polynomials are chosen because of their elegant properties, such as an efficient even distribution of error that leads to rapid convergence, minimization of the maximum error, and economization of the power series.

An interpolation of Equation (3) at point $x=x_n$ and collocation of its 1st and 2nd derivatives at point $x=x_{n+v}$ yields a system of equations. Thus, we have

$$\sum_{r=3}^{s+9}(r^3-3r^2+2r)\sigma_r X_{n+w}^{r-3}=k_{n+w}\,.$$

where k_{n+w} denotes the third derivative of Equation (3).

The interpolation and collocation of (3) produce a system of equations represented as

$$\begin{cases}\displaystyle\sum_{r=0}^{s+9}\sigma_r X_n^r=y_n\\ \displaystyle\sum_{r=1}^{s+9}r\sigma_r X_{n+v}^{r-1}=m_{n+v}\\ \displaystyle\sum_{r=2}^{s+9}(r^2-r)\sigma_r X_{n+v}^{r-2}=l_{n+v}\\ \displaystyle\sum_{r=3}^{s+9}(r^3-3r^2+2r)\sigma_r X_{n+w}^{r-3}=k_{n+w}\end{cases} \tag{4}$$

where m_{n+v} is the first derivative of (3), l_{n+v} is its second derivative, and s is the step number (s = 2).

The proposed method is formulated by specifying parameters v and w. Thus, $v=0,\frac{2}{8},\frac{5}{8},1$ and $w=2$.

By solving the system of ten equations for $\sigma's$ using MAPLE and substituting the resulting values into Equation (3), the continuous implicit form of THBM is obtained as

$$\sigma_0(t)y_n = h\left(\sum_{r=0}^{2}\beta_1(t)m_{n+1} + \beta_{\frac{2}{8}}(t)m_{n+\frac{2}{8}} + \beta_{\frac{5}{8}}(t)m_{n+\frac{5}{8}}\right) + h^2\left(\sum_{r=0}^{2}\mu_1(t)l_{n+1} + \mu_{\frac{2}{8}}(t)l_{n+\frac{2}{8}} + \mu_{\frac{5}{8}}(t)l_{n+\frac{5}{8}}\right) + h^3\left(\delta_2(t)k_{n+2}\right) \tag{5}$$

where, $t = \dfrac{2x - 2x_n - h}{h}$, $\sigma_0(t) = 1$.

$$\begin{cases}
\alpha_0 - \beta_0 + \beta_{\frac{2}{8}} - \beta_{\frac{5}{8}} + \beta_1 - \lambda_0 + \lambda_{\frac{2}{8}} - \lambda_{\frac{5}{8}} + \lambda_1 - \delta_2 = y_n \\
2\beta_0 - 8\beta_{\frac{2}{8}} + 18\beta_{\frac{5}{8}} - 32\beta_1 + 50\lambda_0 - 72\lambda_{\frac{2}{8}} + 98\lambda_{\frac{5}{8}} - 128\lambda_1 + 162\delta_2 = hm_n \\
2\beta_0 - 4\beta_{\frac{2}{8}} + 0\beta_{\frac{5}{8}} + 8\beta_1 - 10\lambda_0 + 0\lambda_{\frac{2}{8}} + 14\lambda_{\frac{5}{8}} -`16\lambda_1 + 0\delta_2 = hm_{n+\frac{2}{8}} \\
2\beta_0 + 8\beta_{\frac{2}{8}} + 18\beta_{\frac{5}{8}} + 32\beta_1 + 50\lambda_0 + 72\lambda_{\frac{2}{8}} + 98\lambda_{\frac{5}{8}} + 128\lambda_1 + 162\delta_2 = hm_{n+\frac{5}{8}} \\
2\beta_0 + 2\beta_{\frac{2}{8}} - \frac{9}{2}\beta_{\frac{5}{8}} - 7\beta_1 + \frac{25}{8}\lambda_0 + \frac{99}{8}\lambda_{\frac{2}{8}} + \frac{91}{32}\lambda_{\frac{5}{8}} -`\frac{119}{8}\lambda_1 - \frac{1539}{128}\delta_2 = hm_{n+1} \\
16\beta_{\frac{2}{8}} - 96\beta_{\frac{5}{8}} + 320\beta_1 - 800\lambda_0 + 1680\lambda_{\frac{2}{8}} - 3136\lambda_{\frac{5}{8}} + 5376\lambda_1 - 8640\delta_2 = h^2 l_n \\
16\beta_{\frac{2}{8}} - 48\beta_{\frac{5}{8}} + 32\beta_1 - 80\lambda_0 - 192\lambda_{\frac{2}{8}} + 122\lambda_{\frac{5}{8}} + 192\lambda_1 - 432\delta_2 = h^2 l_{n+\frac{2}{8}} \\
16\beta_{\frac{2}{8}} + 24\beta_{\frac{5}{8}} - 40\beta_1 - 100\lambda_0 + 15\lambda_{\frac{2}{8}} + \frac{413}{2}\lambda_{\frac{5}{8}} + 111\lambda_1 - 270\delta_2 = h^2 l_{n+\frac{5}{8}} \\
16\beta_{\frac{2}{8}} + 96\beta_{\frac{5}{8}} + 320\beta_1 + 800\lambda_0 + 1680\lambda_{\frac{2}{8}} + 3136\lambda_{\frac{5}{8}} + 5376\lambda_1 + 8640\delta_2 = h^2 l_{n+1} \\
192\beta_{\frac{5}{8}} + 4608\beta_1 + 68160\lambda_0 + 801792\lambda_{\frac{2}{8}} + 8227968\lambda_{\frac{5}{8}} + 77064192\lambda_1 + 675946368\delta_2 = h^3 k_{n+2}
\end{cases}$$

Equation (5) is evaluated at $x = x_{n+1}(t=1)$, $x = x_{n+\frac{1}{4}}(t = \frac{-1}{2})$, $x = x_{n+\frac{5}{8}}(t = \frac{1}{4})$ and $x = x_{n+2}(t=3)$. This yields the following formulae:

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$$\left.\begin{aligned}
y_{n+\frac{1}{4}} &= y_n - \frac{88137791}{7502028192} l_{n+\frac{1}{4}} h^2 - \frac{2444852359}{16804543150080} l_{n+1} h^2 - \frac{854061367}{410267166750} l_{n+\frac{5}{8}} h^2 \\
&\quad + \frac{864884159429}{8644312320000} h m_n + \frac{25652175317}{16804543150080} m_{n+1} h + \frac{9108020977}{65642746680} m_{n+\frac{1}{4}} h \\
&\quad + \frac{1101992704}{113963101875} m_{n+\frac{5}{8}} h + \frac{607}{59275284480} k_{n+2} h^3 + \frac{855155449}{288143744000} l_n h^2 \\
y_{n+\frac{5}{8}} &= y_n - \frac{513919379375}{860392609284096} l_{n+1} h^2 + \frac{5112396875}{960259608576} l_{n+\frac{1}{4}} h^2 - \frac{13252571975}{840227157504} l_{n+\frac{5}{8}} h^2 \\
&\quad + \frac{418129035395}{3540710326272} m_n h + \frac{5667019294375}{860392609284096} m_{n+} h + \frac{1048820088125}{3360908630016} m_{n+\frac{1}{4}} h \\
&\quad + \frac{274615925}{1458727704} m_{n+\frac{5}{8}} h + \frac{100625}{3034894565376} k_{n+2} h^3 + \frac{604142225}{147529596928} l_n h^2 \\
y_{n+1} &= y_n - \frac{224396423}{32821373340} l_{n+1} h^2 + \frac{4552256}{234438381} l_{n+\frac{1}{4}} h^2 + \frac{3855855616}{205133583375} l_{n+\frac{5}{8}} h^2 \\
&\quad + \frac{610085017}{4220855625} m_n h + \frac{1241893456}{8205343335} m_{n+1} h + \frac{2815556096}{8205343335} m_{n+\frac{1}{4}} h \\
&\quad + \frac{41137340416}{113963101875} m_{n+\frac{5}{8}} h + \frac{7}{57886020} k_{n+2} h^3 + \frac{6617941}{1125561500} l_n h^2 \\
y_{n+2} &= y_n - \frac{1605440031568}{8205343335} m_{n+1} h + \frac{1199003656192}{8205343335} m_{n+\frac{1}{4}} h + \frac{30183451648}{234438381} l_{n+\frac{1}{4}} h^2 \\
&\quad + \frac{202566136196}{8205343335} l_{n+1} h^2 + \frac{83816}{14471505} k_{n+2} h^3 - \frac{8833629945856}{37987700625} m_{n+\frac{5}{8}} h \\
&\quad + \frac{1199028359234}{4220855625} m_{n+\frac{1}{4}} h + \frac{5631909658}{281390375} l_n h^2 + \frac{27285660434432}{205133583375} l_{n+\frac{5}{8}} h^2
\end{aligned}\right\} \qquad (6)$$

which constitutes the proposed THBM.

3. Convergence Analysis of THBM

The convergence analysis of THBM is presented in this section. The order, error constant, zero-stability, and consistency of THBM are all investigated.

Error Constant of THBM

The derived scheme belongs to the class of linear multistep methods of the form:

$$\sum_{r=0}^{s} \sigma_r y_{n+r} = h\left(\sum_{r=0}^{s} \beta_r(t) m_{n+r}\right) + h^2\left(\sum_{r=0}^{s} \mu_r(t) l_{n+r}\right) + h^3\left(\sum_{r=0}^{s} \delta_r(t) k_{n+r}\right) \tag{7}$$

Theorem 3.1

The local truncation error (LTE) of THBM is $C^{10}h^{10}y^{10}(x_n) + O(h^{11})$.

Proof

The LTE associated with Equation (7) is defined by the operator

$$LTE = \sum_{r=0}^{k}\left[\sigma_r y(x_n + rh) - h^2\beta_r m(x_n + rh) - h^3 \gamma_r l(x_n + rh)\right] \tag{8}$$

The $y(x)$ in (8) is an arbitrary function. Equation (8) is expanded using the Taylor series about the point x, giving the expression below.

$$LTE = \lambda_0 y(x) + \lambda_1 h y'(x) + \lambda_2 h^2 y''(x) + \ldots + \lambda_{p+3} h^{p+3} y^{p+3}(x)$$

where the λ_0, $\lambda_1, \lambda_2 \ldots \lambda_p \ldots \lambda_{p+2}$ are obtained as $\lambda_0 = \sum_{r=0}^{s}\sigma_r$, $\lambda_1 = \sum_{r=1}^{s} r\sigma_r$, $\lambda_2 = \frac{1}{2!}\sum_{r=1}^{s} r^2\sigma_r$, $\lambda_q = \frac{1}{q!}\left[\sum_{r=1}^{s} r^q\alpha_r - q(q-1)\sum_{r=1}^{s}\beta_r r^{q-2} - q(q-1)(q-2)\sum_{r=1}^{s}\gamma_r r^{q-3}\right]$.

According to [16], Equation (8) is of order p if $\lambda_0 = \lambda_1 = \lambda_2 = \ldots \lambda_p = \lambda_{p+1} = 0$ and $\lambda_{p+j} \neq 0$. The $\lambda_{p+j} \neq 0$ is called the error constant, and $\lambda_{p+j} h^{p+j} y^{p+j}(x_n)$ is the principal local truncation error at the point x_n.

Thus, with further simplification of the above equation, the order of THBM is obtained as $p = 7$ with an error constant

$$\lambda_{10} = \left[\frac{16772791859}{4111529482232266752 00}, \frac{500772879875}{421020618980584115 4048}, \frac{17950369}{501895688749056 00}, \frac{17378108219}{313684805468 1600}\right]^T$$

.

Zero-Stability of the THBM

This property concerns the ability of the proposed numerical method to remain bounded and avoid the exponential growth of errors when applied to differential equations with zero initial conditions.

Equation (7) is presented below in the form of column vectors in order to analyze the zero-stability of the scheme:

$\varepsilon = (\varepsilon_1 \ldots \varepsilon_r)^T$, $d = (d_1 \ldots d_j)^T$, $y_m = (y_{n+1} \ldots y_{n+j})^T$, $M(y_m) = (m_{n+1} \ldots m_{n+j})^T$,$L(y_m) = (l_{n+1} \ldots l_{n+j})^T$, $K(y_m) = (k_{n+1} \ldots k_{n+j})^T$ and matrices $A = (a_{ij})$, $B = (b_{ij})$.

Thus, Equation (6) is presented as

$$A^0 y_m = hRV(y_m) + A^1 y_n + hrv_n + h^2 UL(y_m) + h^2 Ul_n + h^3 MW(y_m) + h^3 uK_n \tag{9}$$

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where h is the fixed mesh size within a block.

In line with Equation (9),

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$b = \begin{pmatrix} \frac{864884159429}{8644312320000} \\ \frac{9108020977}{65642746680} \\ \frac{1101992704}{113963101875} \\ \frac{25652175317}{16804543150080} \\ 0 \end{pmatrix}, \quad d = \begin{pmatrix} \frac{855155449}{288143744000} \\ \frac{604142225}{147529596928} \\ \frac{6617941}{1125561500} \\ \frac{5631909658}{281390375} \end{pmatrix}, \quad u = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{607}{59275284480} \end{pmatrix},$$

$$B = \begin{pmatrix} \frac{418129035395}{3540710326272} & \frac{610085017}{4220855625} & \frac{1199028359234}{4220855625} \\ \frac{1048820088125}{3360908630016} & \frac{2815556096}{8205343335} & \frac{1199003656192}{8205343335} \\ \frac{274615925}{10714587277041} & \frac{41137340416}{113963101875} & \frac{-8833629945856}{37987700625} \\ \frac{5667019294375}{860392609284096} & \frac{1241893456}{8205343335} & \frac{-1605440031568}{8205343335} \\ 0 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} \frac{604142225}{147529596928} & \frac{6617941}{1125561500} & \frac{5631909658}{281390375} \\ \frac{5112396875}{960259608576} & \frac{4552256}{234438381} & \frac{30183451648}{234438381} \\ \frac{-13252571975}{840227157504} & \frac{3855855616}{205133583375} & \frac{27285660434432}{205133583375} \\ \frac{-513919379375}{860392609284096} & \frac{-224396423}{32821373340} & \frac{202566136196}{8205343335} \\ 0 & 0 & 0 \end{pmatrix} \text{ and}$$

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{100625}{3034894565376} & \frac{7}{57886020} & \frac{6875}{33312384} & \frac{83816}{14471505} \end{pmatrix}$$

Equation (10) is the first characteristic polynomial of THBM, given by

$$\rho(R) = \det(RA^0 - A^1) \tag{10}$$

where

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } A^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Substituting A^0 and A^1 in Equation (10) and solving for R, the values of R are obtained as 0, 0, 0, and 1.

According to Lambert (1991), the zero-stability of THBM is guaranteed since $|R_j| \le 1$, $j = 1$ is satisfied by $\rho(R) = 0$ and the multiplicity of the roots with $|R_j| = 1$ does not exceed three.

Consistency of THBM

By introducing the exact solution into the proposed THBM and requiring that the residual vanishes as the step size (h) approaches zero, the consistency of the model is examined. The order of consistency, established above to be seven, is determined by the rate at which the residual tends to zero.

An LMM must be zero-stable and consistent in order to be convergent (Dahlquist, 1979). These properties confirm the convergence of THBM.

4. Numerical Experiments

In this section, the accuracy of THBM is examined using five test problems. Example 3, a second-order initial value problem (IVP), and Example 4, a third-order IVP, presented in Adeyefa (2020) and Kuboye (2015), respectively, were solved using the Second- and Third-Order Model (STOM) and the Single-Order Model (SOM). Similarly, Examples 2 and 5, both third-order problems from Kuboye (2015), were solved using SOM. In addition, the solutions to these examples were obtained using the First-, Second-, and Third-Order Model (FSTOM) and the Third-Order Integrator (TOI), respectively. Test Problem 1, a first-order epidemiological model, was solved in Adeyefa (2020) and Ajileye et al. (2018) using FSTOM and the First-Order Model (FOM), respectively.

THBM is compared with all of these existing methods, and efficiency curves are presented for each problem. These curves plot the logarithm of the maximum error against the number of function evaluations. The comparative analysis and efficiency curves provide useful insights into the performance and accuracy of THBM relative to other numerical methods for the test problems considered.

Test Problems (TP)

TP 1: The *epidemiological model* **known as the** *SIR model is considered here, where* $S(t)$ *is the number of susceptible people,* $I(t)$ *is the number of people infected and* $R(t)$ *is the number of people who have recovered. It is given by the following three coupled equations:*

$$\frac{dS}{dt} = \mu(1 - S) - \beta IS \tag{i}$$

$$\frac{dI}{dt} = -\mu I - \gamma I + \beta IS \tag{ii}$$

$$\frac{dR}{dt} = -\mu R + \gamma I \tag{iii}$$

where μ, γ and β are positive parameters and y is defined as

$$y = S + I + R \tag{iv}$$

Adding Equations (i), (ii), and (iii), the evolution equation for y is obtained as

$$y' = \mu(1 - y) \tag{v}$$

Taking μ and y as 0.5 each, the IVP

$$y' = \frac{1-y}{2}, \quad y(0) = 0.5, h = 0.1 \tag{vi}$$

whose exact solution is

$$y(t) = 1 - \frac{1}{2}e^{-0.5t}, \tag{vii}$$

is obtained.

Table 1. TP 1 Errors Comparison

X	THBM	[2]	[3]
0.1	1.497820E-08	1.0E-10	1.218026E-13
0.2	2.849542E-08	1.0E-10	1.399991E-13
0.3	4.065853E-08	1.0E-10	1.184941E-12
0.4	5.156745E-08	2.0E-10	1.538991E-12
0.5	6.131560E-08	3.0E-10	1.110001E-12
0.6	6.999025E-08	3.0E-10	5.270229E-12
0.7	7.767291E-08	2.0E-10	2.10898E-12
0.8	8.443973E-08	3.0E-10	1.297895E-11
0.9	9.036176E-08	3.0E-10	3.08229E-11
1.0	9.550529E-08	2.0E-10	4.121925E-11

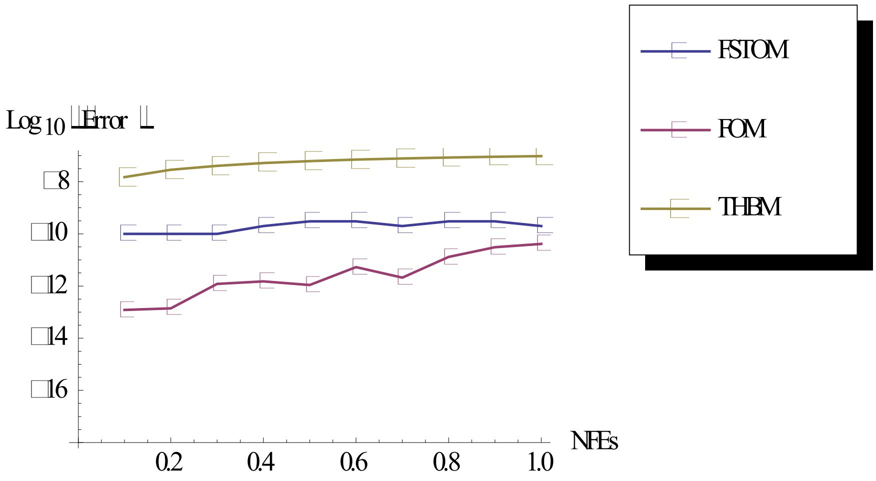

Figure 1. Efficiency Curves for TP 1

This problem is a *first-order epidemiological model solved using* FSTOM and FOM. The proposed THBM was also applied to TP1. Figure 1 shows that the two existing methods are more accurate, but they cannot handle multi-order ODEs.

TP 2: $$\frac{d^3y}{dx^3} = 3\sin x, \quad y(0) = 1, y'(0) = 0, y''(0) = -2, h = 0.1$$

Analytical Solution: $y(x) = 3\cos x + \frac{x^2}{2} - 2$

Table 2. TP 2 Errors Comparison

x- values	THBM	[2]	[14]
0.1	1.070284E-19	2.000000E-010	6.370460E-13
0.2	2.127975E-19	4.000000E-010	4.052980E-12
0.3	3.162495E-19	2.000000E-010	1.009326E-11
0.4	4.163510E-19	2.000000E-010	1.890366E-11
0.5	5.121032E-19	9.000000E-010	3.033807E-11
0.6	6.025481E-19	1.100000E-009	4.455258E-11
0.7	6.867816E-19	1.500000E-009	5.987466E-11
0.8	7.639632E-19	1.300000E-009	7.711903E-11
0.9	8.333217E-19	1.500000E-009	9.618412E-11
1.0	8.941628E-19	2.000000E-009	1.171654E-10

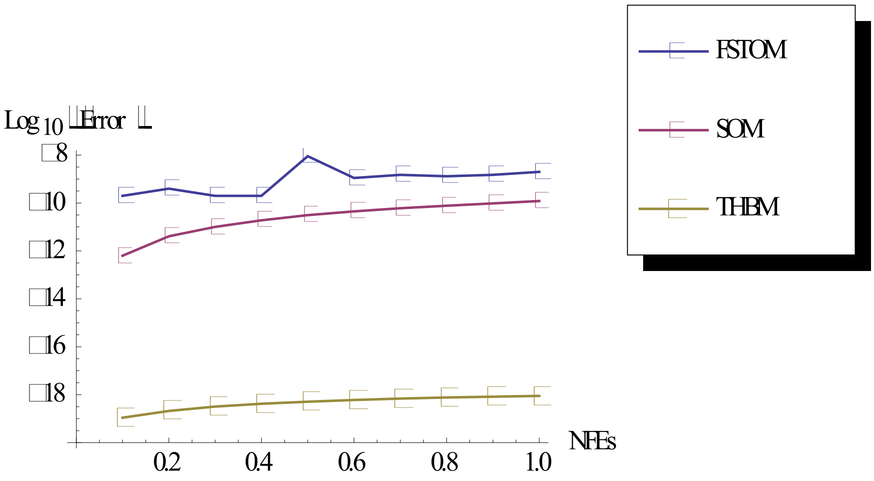

Figure 2. Efficiency Curves for TP 2

TP2 is a third-order IVP. Table 2 compares the errors of THBM with those of the existing methods (FSTOM and SOM), and the efficiency curve is displayed in Figure 2. THBM is clearly more accurate.

TP 3 : $y'' = y', \quad y(0) = 0, y'(0) = -1, \ h = 0.1$

True Solution: $y(x) = 1 - e^{x}$

Table 3. TP 3 Errors Comparison

x- values	THBM	[1]	[14]
0.1	2.649757E-10	2.095826E-10	2.508826E-13
0.2	5.352775E-10	2.092718E-09	6.493175E-11
0.3	8.109857E-10	7.842546E-09	1.683146E-09
0.4	1.092181E-09	2.009500E-08	1.700635E-08
0.5	1.378947E-09	4.199771E-08	1.025454E-07
0.6	1.671367E-09	7.728842E-08	2.558711E-06
0.7	1.969526E-09	1.303844E-07	5.273300E-06
0.8	2.273509E-09	2.064839E-07	8.275935E-06
0.9	2.583402E-09	3.116817E-07	1.161667E-05
1.0	2.899296E-09	4.531001E-07	1.542187E-05

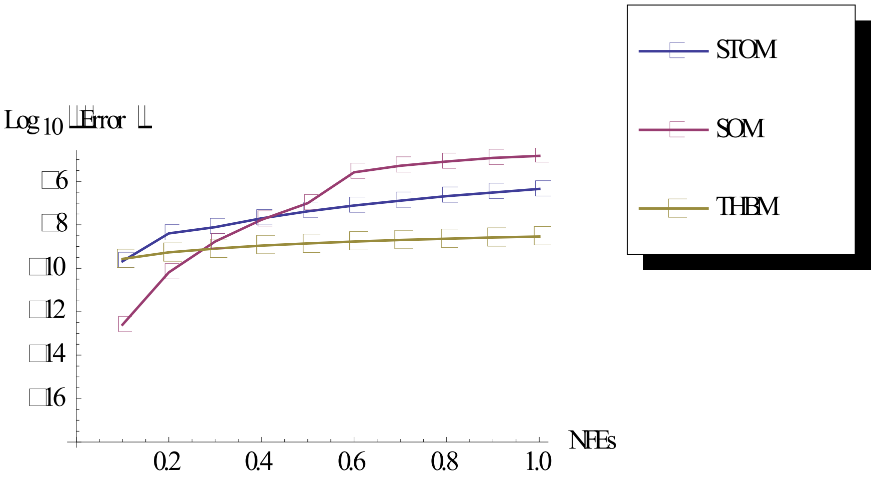

Figure 3. Efficiency Curves for TP 3

TP3 is a second-order IVP. Table 3 compares the errors of THBM with those of the existing methods (STOM and SOM), and the efficiency curve is displayed in Figure 3. The superiority of THBM in terms of accuracy is clear.

TP 4: $\frac{d^3z}{dx^3} = e^x \quad z(0) = 3, z'(0) = 1, z''(0) = 5, h = 0.1$

True Solution: $z(x) = 2(1 + x^2) + e^x$

Table 4. TP 4 Errors Comparison

x- value	THBM	[1]	[14]
0.1	3.827E-20	8.881E-015	3.369E-12
0.2	8.058E-20	3.552E-014	2.160E-11
0.3	1.273E-19	8.304E-014	5.333E-11
0.4	1.790E-19	1.527E-013	9.988E-11
0.5	2.361E-19	2.460E-013	1.598E-10
0.6	2.992E-19	3.668E-013	2.511E-10
0.7	3.689E-19	5.178E-013	3.961E-10
0.8	4.460E-19	7.025E-013	5.926E-10
0.9	5.312E-19	9.254E-013	8.429E-10
1.0	6.254E-19	1.187E-012	1.144E-09

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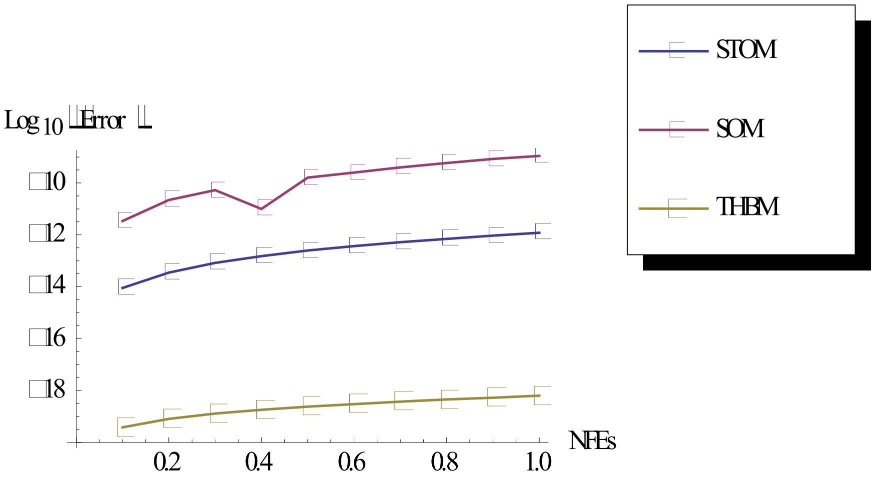

Figure 4. Efficiency Curves for TP 4

TP4 is a third-order IVP. Table 4 compares the errors of THBM with those of the existing methods (STOM and SOM), and Figure 4 shows the efficiency curve. THBM is again shown to be superior in terms of accuracy.

TP 5 : $y''' = y'(2xy''+y'), \quad y(0)=1, y'(0)=0.5, y''(0)=0, \; h=0.01.$

True Solution: $y(x)=1+\frac{1}{2}In\left(\frac{2+x}{2-x}\right).$

Table 5: TP 5 Errors Comparison

x- values	THBM	[14]	[8]
0.01	0.000000E+000	2.508826E-13	1.194048E-013
0.02	0.000000E+000	6.493175E-11	4.086842E-013
0.03	1.000000E-024	1.683146E-09	1.016689E-012
0.04	0.000000E+000	1.700635E-08	2.139484E-012
0.05	0.000000E+000	1.025454E-07	4.083580E-012
0.06	1.000000E-024	2.558711E-06	7.350069E-012
0.07	0.000000E+000	5.273300E-06	1.279204E-011

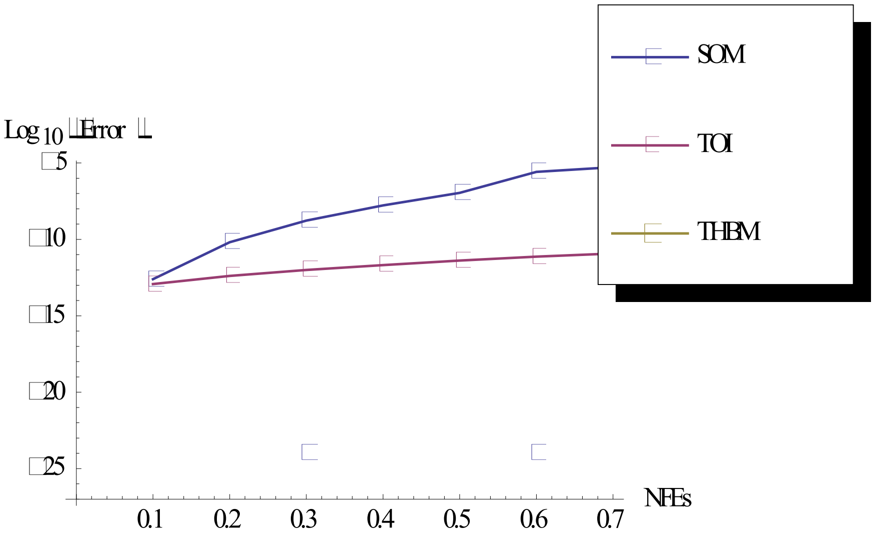

Figure 5. Efficiency Curves for TP 5

TP5 is a third-order IVP. Table 5 and Figure 5 show that THBM reproduces the exact results.

Discussion of Results

Overall, Tables I to V display the numerical solutions obtained from the application of THBM.

TP1 is a first-order ODE, TP3 is a second-order ODE, and TP2, TP4, and TP5 are third-order ODEs. THBM solved all of these test problems. Comparison with existing methods shows that THBM consistently delivers favorable performance, as it handles three different orders compared with the single order of the existing methods. Also, there is a reduction in error in the majority of the cases considered, with the exception of TP1, where the existing methods compared are more accurate but are limited to a single order. These tabular results provide strong evidence of the capabilities of THBM in solving first-, second-, and third-order ODEs.

5. Conclusion

This study focuses on the direct solution of first-, second-, and third-order ordinary differential equations (ODEs) using THBM. With its eighth-order accuracy, THBM is shown to be consistent. Its capability to solve differential equations of various orders provides a significant advantage over existing numerical methods. Several ODEs of orders 1, 2, and 3 are used to illustrate the efficacy of the proposed methodology. The results, presented in Tables I to V, surpass those obtained by existing methods in terms of accuracy. This demonstrates the superior performance of THBM in tackling ODEs of multiple orders and supports its potential as an advanced numerical solver in various scientific and engineering applications. Although THBM cannot handle ODEs of orders higher than three, the approach can be extended to solve stiff IVPs and partial differential equations.

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